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STABILITY CONDITION FOR ARAKAWA' S *B* GRID

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ABSTRACT

Most models that demand a coarse grid resolution use Arakawa' s B grid. Using this particular grid, it is shown that the CFL condition, for the fastest travelling wave, is twice the value than for both the non-staggered and the staggered C grid, respectively.

RESUMEN

La mayoría de los modelos que requieren poca resolución horizontal utilizan la grilla B de Arakawa. Mediante la técnica de estabilidad lineal, demostraremos que la condición de estabilidad para la onda de gravedad es el doble de la requerida para las grillas A y C de Arakawa. La onda de gravedad es la de mayor velocidad de propagación en el modelo de aguas someras.

1. INTRODUCTION

Bryan (1969), Gill and Bryan (1971), build up their numerical models, employing Arakawa' s B lattice, where the free surface elevation is evaluated at the center of the grid, while both velocity components, u and v, are evaluated at the four corners of the grid (Fig. 1). In computing both, the Coriolis and the nonlinear terms, quite an amount of averaging is omitted.

The theoretical analysis of Mesinger and Arakawa (1976) strongly suggests that grid C produce better numerical simulations than grid B. This situation holds true as long as the gridpoint resolution is smaller than the Rossby radius of deformation. The reverse is also true. Whenever the Rossby radius of deformation is smaller than the gridpoint size, grid B gives better numerical simulations than grid C (Bryan, 1989; page 473). Most numerical models that demand a coarse-grid resolution are designed using the B grid (Semtner, 1986).

The aim of this study is to gain some understanding as to why lattice B works better for numerical models with coarse-grid resolution. Following the pioneering work of

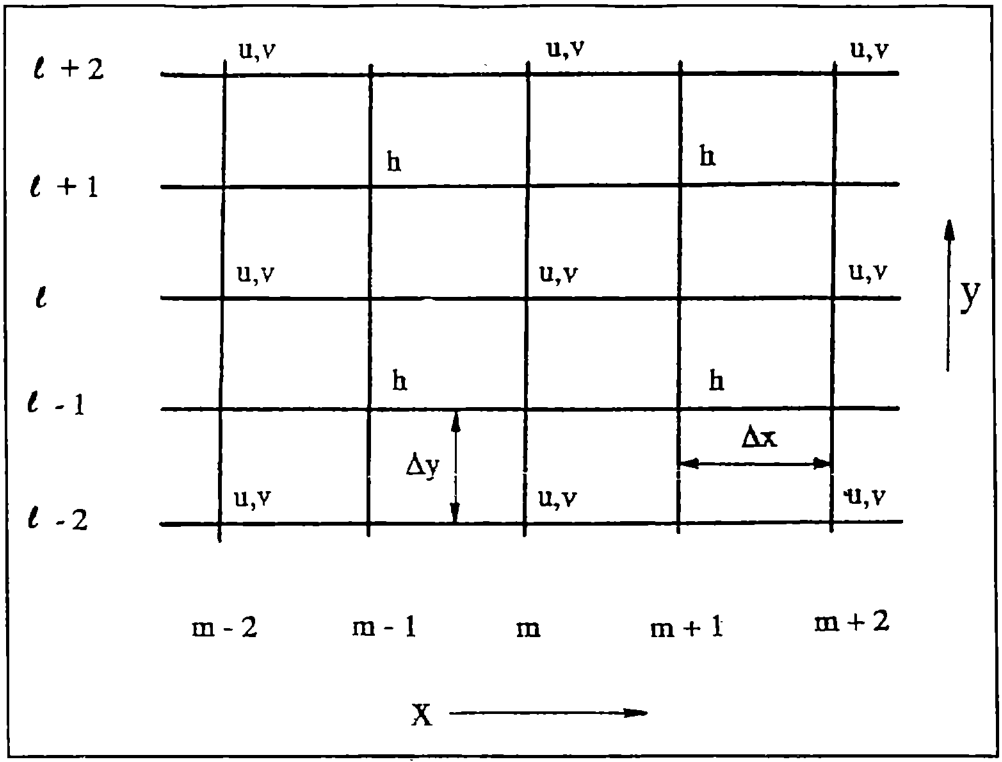

Fig. 1. Spatial representation of Arakawa' s B lattice.

O'Brien (1986) on stability conditions, a series of analytical studies is conducted. For this purpose, the linear stability technique developed by von Neumann is used (Charney et al., 1950).

In our analysis, we choose to index all the gridpoints and consider the distance between grid points as Δx and Δy; where Δx and Δy, represent the gridpoint resolution in the east-west and north south directions, respectively (Fig. 1). Other investigators choose to call the distance between the same variable as the grid resolution. Our indexation makes the interpretation of the stability results more interesting. The same indexation criterion was followed by O' Brien (1986) and O' Brien & Inoue (1982) (Fig. 2). The results are quite conclusive. It is shown that the stability condition for the gravity (fastest travelling) wave is twice the value than for the non-staggered and the C grids, respectively (Mesinger & Arakawa, 1976; O' Brien, 1986; O' Brien & Inoue, 1982).

2. THE PROBLEM

The general stability condition of the finite difference scheme is determined by the general condition (Mesinger & Arakawa, 1976):

$$C \Delta t / \Delta x \leq O(1)$$

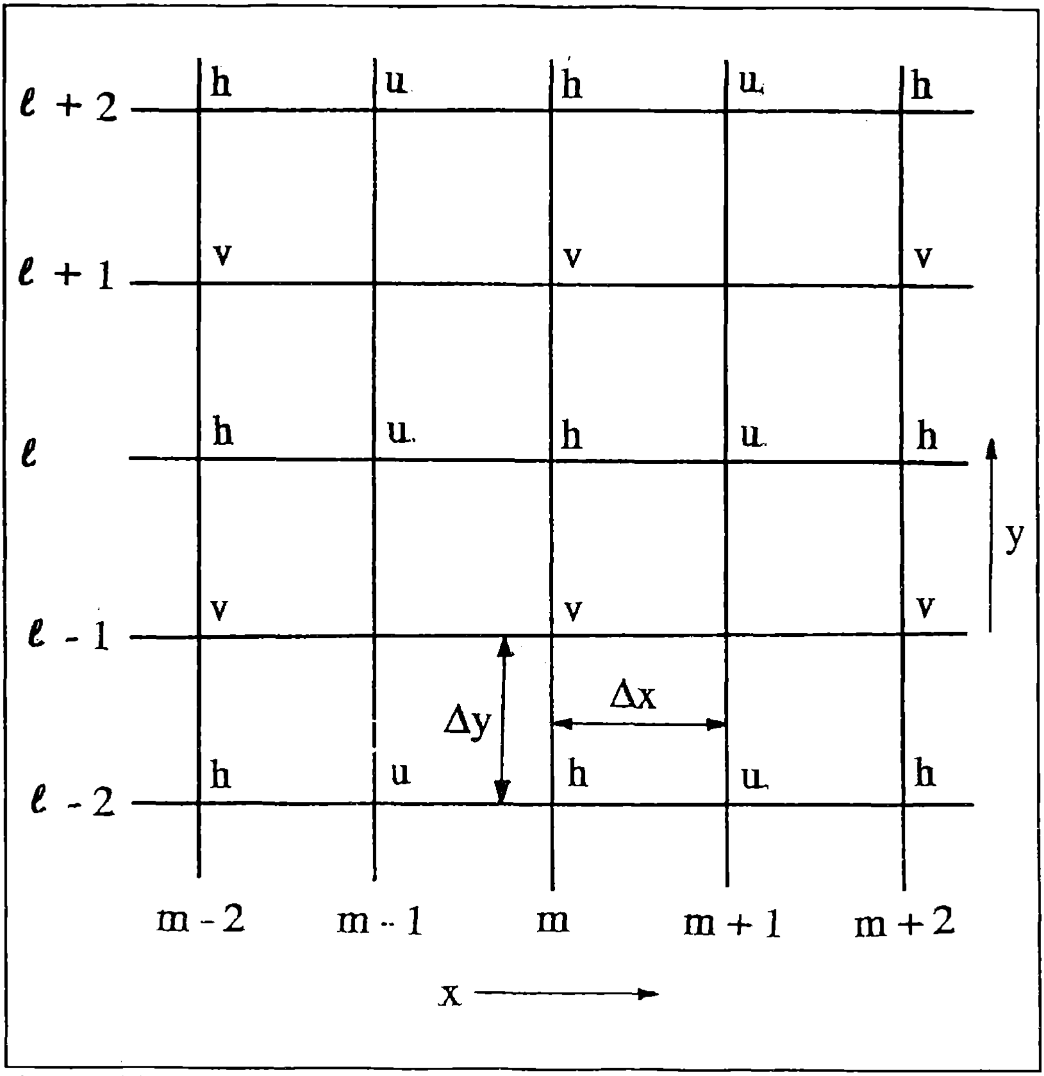

Fig. 2. Spatial representation of Arakawa' s C lattice.

3. STABILITY ANALYSIS

3.1. Analysis for the one - dimensional gravity wave

Consider the following set of partial differential equations:

$$\begin{aligned} \partial u / \partial t &= - g\ \partial h / \partial x \\ \partial h / \partial t &= - H\ \partial u / \partial x, \end{aligned} \tag{1}$$

where g represents the earth' s gravity acceleration; u and v, the velocity components in the east-west and north-south directions, respectively; h, the free-surface elevation; and H the mean sea level depth.

A second order, centered in space and time finite difference scheme is employed. It is obtained:

$$\begin{aligned} u^{n+1}_{m,l} &= u^{n-1}_{m,l} - (g\gamma/2)[(h^{n}_{m+1,l+1} + h^{n}_{m+1,l-1}) - (h^{n}_{m-1,l+1} + h^{n}_{m-1,l-1})] \\ h^{n+1}_{m+1,l+1} &= h^{n-1}_{m+1,l+1} - (H\gamma/2) [(u^{n}_{m+2,l+2} + u^{n}_{m+2,l}) - (u^{n}_{m,l+2} + u^{n}_{m,l})] \end{aligned} \tag{2}$$

where the superscript *n*, stands for time level; the subscripts *(m,l)*, the mesh of discrete points in the x and y directions, respectively; Δx and Δy, the gridpoint resolution in the x and y directions, respectively; γ is equal to $\Delta t / \Delta x$, and Δt represents the time step increment (Fig.1).

Define $C^2 = g H$, $\theta\ (= \mu \Delta x)$ and $\sigma\ (= \nu \Delta y)$, where μ and ν, are the east-west and north-south wavenumbers, respectively. It is convenient to define $Q = (u,v,h)$. Assume:

$$Q^{n}_{m,l} = Q_n \exp (i\, m\, \mu \Delta x) \exp (i\, l\, \nu \Delta y) \tag{3}$$

Upon substitution of equation (3) into the set of equations (2), yields:

$$\begin{aligned} u_{n+1} &= u_{n-1} - \gamma g (2 i \sin \theta \cos \sigma) h_n \\ h_{n+1} &= h_{n-1} - \gamma H (2 i \sin \theta \cos \sigma) u_n \end{aligned} \tag{4}$$

It is convenient to define an amplification factor, Z, such that:

$$Q_{n+2} = Z\, Q_n \tag{5}$$

In doing so, equations (4) may be rewritten as:

$$\begin{aligned} L_1 u_n \quad &+ \quad L_2 h_n = 0 \\ L_1 h_n \quad &+ \quad L_3 u_n = 0 \end{aligned} \tag{6}$$

The operators L_1, L_2 and L_3 are defined as:

$$\begin{aligned} L_1 &= Z^{1/2} - Z^{-1/2} \\ L_2 &= 2\, i\, g\, \gamma \sin\theta \cos\sigma \\ L_3 &= 2\, i\, H\, \gamma \sin\theta \cos\sigma \end{aligned} \tag{7}$$

The homogeneous set of equations (6) is solved by letting:

$$L_1^2 - L_2 L_3 = 0. \tag{8}$$

A second order equation for Z is obtained. Namely:

$$Z^2 - 2\,(1 - 2\,C^2 \gamma^2 \sin^2\theta \cos^2\sigma)\, Z + 1 = 0. \tag{9}$$

Two complex conjugate solutions are obtained

$$\begin{aligned} Z_- &= G - i\,(1 - G^2)^{1/2} \\ Z_+ &= G + i\,(1 - G^2)^{1/2} \end{aligned} \tag{10}$$

where

$$G = 1 - 2\,(C\,\gamma \sin\theta \cos\sigma)^2 \tag{11}$$

To have a stable (neutral) condition the absolute value of Z should be less (equal) than (to) one. Otherwise, the finite difference scheme under consideration will be unstable. In multiplying the two solutions yields:

$$|\, Z\,|^2 = Z_+ \, Z_- = 1 \tag{12}$$

Therefore, if the term under the radical sign, $1 - G^2$, is positive, then the absolute value of the amplification factor will be equal to one. This instance will hold certain if, and only if:

$$(C\,\gamma \sin\theta \cos\sigma)^2 \leq 1. \tag{13}$$

Given the fact that the absolute value of the sine and the cosine is less or equal to unity, and recalling the definition of γ, it follows that:

$$C\,\Delta t / \Delta x \le 1, \tag{14}$$

which is the classical C-F-L condition for computational stability.

3.2. Stability analysis for the inertial gravity wave

Let us consider the following set of partial differential equations:

$$\begin{aligned} \partial u / \partial t &= f v - g\, \partial h / \partial x \\ \partial v / \partial t &= - f u \\ \partial h / \partial t &= - H\, \partial u / \partial x, \end{aligned} \tag{15}$$

Upon using the same finite difference scheme as above yields:

$$\begin{aligned} u_{n+1} &= u_{n-1} + 2\, f \Delta t\, v_n - \gamma\, g\,(2\, i \sin\theta \cos\sigma)\, h_n \\ v_{n+1} &= v_{n-1} - 2\, f \Delta t\, u_n \\ h_{n+1} &= h_{n-1} - \gamma\, H\,(2\, i \sin\theta \cos\sigma) u_n \end{aligned} \tag{16}$$

Equations (16) may then be rewritten as:

$$\begin{aligned} L_1 u_n - L_4 v_n + L_2 h_n &= 0 \\ L_1 v_n + L_4 u_n &= 0 \\ L_1 h_n + L_3 u_n &= 0 \end{aligned} \tag{17}$$

where $L_4 = 2 f \Delta t$.

Following the same procedure as above yields:

$$Z^2 - 2\,(1 - 2\, C^2 \gamma^2 \sin^2\theta \cos^2\sigma - 2\,(f \Delta t)^2)\, Z + 1 = 0. \tag{18}$$

The two complex conjugate solutions are:

$$\begin{aligned} Z_- &= G - i\,(1 - G^2)^{1/2} \\ Z_+ &= G + i\,(1 - G^2)^{1/2} \end{aligned} \tag{19}$$

where the value of G has changed. It is:

$$G = 1 - 2(f\Delta t)^2 - 2(C\gamma \sin\theta \cos\sigma)^2 \tag{20}$$

In multiplying the two complex conjugate solutions, as in the previous case, we will arrive to a neutral stability condition, provided that the term under the radical sign, $1 - G^2$, is positive. This will require that:

$$(f\Delta t)^2 + (C\gamma \sin\theta \cos\sigma)^2 \leq 1 \tag{21}$$

Thus, this is the same stability condition as the one for the unstaggered grid case (O' Brien, 1986; page 177):

$$C\gamma \leq [1 - (f\Delta t)^2]^{1/2} \tag{22}$$

3.3. Stability condition for a two dimensional flow

Let us consider the following set of equations:

$$\begin{aligned} \partial u/\partial t &= fv - g\,\partial h/\partial x \\ \partial v/\partial t &= -fu - g\,\partial h/\partial y \\ \partial h/\partial t &= -H(\partial u/\partial x + \partial v/\partial y) \end{aligned} \tag{23}$$

In using a centered in space and time finite differencing scheme, yields:

$$\begin{aligned} u_{n+1} &= u_{n-1} + 2f\Delta t\, v_n - 2i\gamma g \sin\theta \cos\sigma\, h_n \\ v_{n+1} &= v_{n-1} - 2f\Delta t\, u_n - 2i\eta g \cos\theta \sin\sigma\, h_n \\ h_{n+1} &= h_{n-1} - 2i\gamma H \sin\theta \cos\sigma\, u_n - 2i\eta H \cos\theta \sin\sigma\, v_n \end{aligned} \tag{24}$$

where η is equal to $\Delta t / \Delta y$. This system of equations, 24, may be rewritten as:

$$\begin{aligned} L_1 u_n - L_4 v_n + L_2 h_n &= 0 \\ L_1 v_n + L_4 u_n + L_5 h_n &= 0 \\ L_1 h_n + L_3 u_n + L_6 v_n &= 0 \end{aligned} \tag{25}$$

where

$$\begin{aligned} L_5 &= 2i\, g\eta \sin\sigma \cos\theta \\ L_6 &= 2i H\eta \sin\sigma \cos\theta \end{aligned} \tag{26}$$

For this set of equations to have unique solution, it follows that:

$$L_1 (L_1^2 - L_2 L_3 - L_5 L_6 + L_4^2) = 0 \tag{27}$$

Following the same procedure as in the two previous cases, yields:

$$Z^2 - 2 \psi Z + 1 = 0 \tag{28}$$

where

$$\psi = 1 - \{ 2 (f \Delta t)^2 + 2 C^2 (\gamma^2 \sin^2 \theta \cos^2 \sigma + \eta^2 \sin^2 \sigma \cos^2 \theta) \} \tag{29}$$

As before, a neutral stability condition will be obtained if:

$$(f \Delta t)^2 + C^2 (\gamma^2 \sin^2 \theta \cos^2 \sigma + \eta^2 \sin^2 \sigma \cos^2 \theta) \leq 1 \tag{30}$$

Several cases are considered:

a) For $L_x = 2 \Delta x$ ($4 \Delta x$) and $L_y = 2 \Delta y$ ($4 \Delta y$), i. e., $\theta = \sigma = \pi$ ($=\pi/2$), where L_x and L_y are the wavelengths in the x and y directions, respectively, yields:

$$|f \Delta t| \leq 1 \qquad \text{for stability} \tag{31}$$

This same result holds for very long waves. Namely for $\theta = \sigma = 0$.

b) For $L_x = 8 \Delta x$ and $L_y = 8 \Delta y$, i. e., $\theta = \sigma = \pi / 4$, yields:

$$(f \Delta t)^2 + C^2 (\gamma^2 + \eta^2) \leq 1 \tag{32}$$

If $\Delta x = \Delta y = \Delta$, yields:

$$C \Delta t / \Delta \leq \{ 2 (1 - (f \Delta t)^2) \}^{1/2} \tag{33}$$

Following O' Brien (1986), the CFL condition for the 2 D gravity wave for the unstaggered grid, is:

$$C \Delta t / \Delta \leq \{ (1 - (f \Delta t)^2)/2 \}^{1/2} \tag{34}$$

This same result is obtained, for the C grid, using this same type of indexation (O' Brien & Inoue, 1982). Without considering the Coriolis parameter, Mesinger and Arakawa

(1976, page 52) show that the stability criterion for the 2 D dimensional gravity wave, for the unstaggered grid, is:

$$C \Delta t / \Delta \leq \{1/2\}^{1/2} \tag{35}$$

which is similar to O' Brien (1986) and O' Brien & Inoue (1982). Therefore, the CFL condition for the B grid, is less restrictive. The stability condition for the gravity wave is twice the value of the stability condition of both the unstaggered grid and grid C.

In their reduced gravity model, Adamec & O' Brien (1978) used a $\Delta t = 10^4$ sec. The product $f \Delta t$ could be of order one. Therefore, the CFL condition, for the reduced gravity models, could easily be violated, if caution is not taken care.

4. CONCLUSIONS

Stability conditions of a series of problems, for the staggered lattice B, leading to the shallow water wave equations are considered. Comparison with stability conditions with both the unstaggered grid and the staggered grid C are made (Mesinger & Arakawa, 1976; O' Brien, 1986; O' Brien & Inoue, 1982). It is shown that the stability condition is double the value than for the other two grids (equations (33) through (35)) This represents a clear improvement of Mesinger and Arakawa' s (1976) solutions.

Therefore, numerical models that require a coarse-grid resolution are designed using the B lattice. In doing so, more accurate solutions, for a coarse grid resolution, are obtained.

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